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Generalized 2-absorbing submodules

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Abstract

In this paper, we will introduce the concepts of generalized 2-absorbing submodules of modules over a commutative ring as generalizations of 2-absorbing submodules and obtain some related results.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity, $\mathbb{Z}$ and $\mathbb{N}$ will denote respectively the ring of integers and the set of natural numbers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7].

Badawi gave a generalization of prime ideals in [3] and said such ideals 2-absorbing ideals. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, and I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. In [4], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The authors in [6] and [12], extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule*

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of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

The purpose of this paper is to introduce the concepts of generalized 2-absorbing submodules of an R-module M as a generalizations of 2-absorbing submodules of M and investigate some properties of this class of modules.

2 Generalized 2-absorbing submodules

Definition 2.1. We say that a proper submodule N of an R-module M is a *generalized 2-absorbing submodule* or *G2-absorbing submodule* of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $a \in \sqrt{(N :_R m)}$ or $b \in \sqrt{(N :_R m)}$ or $ab \in (N :_R M)$.

Example 2.2. Clearly every 2-absorbing submodule is a G2-absorbing submodule. But the converse is not true in general. For example, the submodule $8\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is a G2-absorbing submodule which is not a 2-absorbing submodule. Also, the submodule $\langle 1/p+\mathbb{Z}\rangle$ of $\mathbb{Z}_{p^\infty}$, where p is a prime number, is a G2-absorbing submodule which is not a 2-absorbing submodule.

Example 2.3. Consider the submodule $N = 0$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{42}$. We have $2.3.\bar{7} = 0$ while $2^i.\bar{7} \neq 0$, $3^j.\bar{7} \neq 0$, and $2.3 \notin (0 :_{\mathbb{Z}} M) = 42\mathbb{Z}$ for all $i, j \in \mathbb{N}$. Thus the submodule N of M, is not G2-absorbing submodule.

Lemma 2.4. Let I be an ideal of R and N be a G2-absorbing submodule of M. If $a \in R$, $m \in M$ and $Iam \subseteq N$, then $a \in \sqrt{(N :_R m)}$ or $I \subseteq \sqrt{(N :_R m)}$ or $Ia \subseteq (N :_R M)$.

Proof. Let $a \notin \sqrt{(N :_R m)}$ and $Ia \not\subseteq (N :_R M)$. Then there exists $b \in I$ such that $ba \notin (N :_R M)$. Now, $bam \in N$ implies that $b \in \sqrt{(N :_R m)}$, since N is a G2-absorbing submodule of M. We have to show that $I \subseteq \sqrt{(N :_R m)}$. Let c be an arbitrary element of I. Thus $(b + c)am \in N$. Hence, either $b + c \in \sqrt{(N :_R m)}$ or $(b + c)a \in (N :_R M)$. If $b + c \in \sqrt{(N :_R m)}$, then by $b \in \sqrt{(N :_R m)}$ it follows that $c \in \sqrt{(N :_R m)}$. If $(b + c)a \in (N :_R M)$, then $ca \notin (N :_R M)$, but $cam \in N$. Thus $c \in \sqrt{(N :_R m)}$. Hence, we conclude that $I \subseteq \sqrt{(N :_R m)}$. □

Lemma 2.5. Let I, J be ideals of R and N be a G2-absorbing submodule of M. If $m \in M$ and $IJm \subseteq N$, then $I \subseteq \sqrt{(N :_R m)}$ or $J \subseteq \sqrt{(N :_R m)}$ or $IJ \subseteq (N :_R M)$.

Proof. Let $I \not\subseteq \sqrt{(N :_R m)}$ or $J \not\subseteq \sqrt{(N :_R m)}$. We have to show that $IJ \subseteq (N :_R M)$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $a \notin \sqrt{(N :_R m)}$ but $aJm \subseteq N$. Now, Lemma 2.4, shows that

$aJ \subseteq (N :_R M)$ and so $(I \setminus \sqrt{(N :_R m)})J \subseteq (N :_R M)$, similarly there exists $b \in J \setminus \sqrt{(N :_R m)}$ such that $Ib \subseteq (N :_R M)$ and also $I(J \setminus \sqrt{(N :_R m)}) \subseteq (N :_R M)$. Thus we have $ab \in (N :_R M)$, $ad \in (N :_R M)$ and $cb \in (N :_R M)$. By $a + c \in I$ and $b + d \in J$ it follows that $(a + c)(b + d)m \in N$. Therefore, $a + c \in \sqrt{(N :_R m)}$ or $b + d \in \sqrt{(N :_R m)}$ or $(a + c)(b + d) \in (N :_R M)$. If $a + c \in \sqrt{(N :_R m)}$, then $c \notin \sqrt{(N :_R m)}$ hence, $c \in I \setminus \sqrt{(N :_R m)}$ which implies that $cd \in (N :_R M)$. Similarly by $(b+d) \in \sqrt{(N :_R m)}$, we can deduce that $cd \in (N :_R M)$. If $(a + c)(b + d) \in (N :_R M)$, then $ab + ad + cb + cd \in (N :_R M)$ and so $cd \in (N :_R M)$. Therefore, $IJ \subseteq (N :_R M)$. □

Theorem 2.6. *Let N be a proper submodule of M. The following statement are equivalent:*

(a) N is a $G2$-absorbing submodule of M;

(b) If $IJL \subseteq N$ for some ideals I, J of R and a submodule L of M, then $I \subseteq \sqrt{(N :_R L)}$ or $J \subseteq \sqrt{(N :_R L)}$ or $IJ \subseteq (N :_R M)$.

Proof. $(a) \Rightarrow (b)$ Let $IJL \subseteq N$ for some ideals I, J of R, a submodule L of M and $IJ \not\subseteq (N :_R M)$. Then by Lemma 2.5, for all $m \in L$ either $I \subseteq \sqrt{(N :_R m)}$ or $J \subseteq \sqrt{(N :_R m)}$. If $I \subseteq \sqrt{(N :_R m)}$, for all $m \in L$ we are done. Similarly if $J \subseteq \sqrt{(N :_R m)}$, for all $m \in L$ we are done. Suppose that $m, m_0 \in L$ are such that $I \not\subseteq \sqrt{(N :_R m)}$ or $J \not\subseteq \sqrt{(N :_R m_0)}$. Thus $J \subseteq \sqrt{(N :_R m)}$ and $I \subseteq \sqrt{(N :_R m_0)}$. Since $IJ(m+m_0) \subseteq N$ we have either $I \subseteq \sqrt{(N :_R m + m_0)}$ or $J \subseteq \sqrt{(N :_R m + m_0)}$. By $I \subseteq \sqrt{(N :_R m + m_0)}$, it follows that $I \subseteq \sqrt{(N :_R m)}$ which is a contradiction, similarly by $J \subseteq \sqrt{(N :_R m + m_0)}$ we get a contradiction. Therefore either $I \subseteq \sqrt{(N :_R L)}$ or $J \subseteq \sqrt{(N :_R L)}$.

$(b) \Rightarrow (a)$ This is obvious. □

Proposition 2.7. Let N be a $G2$-absorbing submodule of an R-module M. Then we have the following.

(a) If K is a submodule of M such that $K \not\subseteq N$, then $(N :_R K)$ is a 2-absorbing primary ideal of R.

(b) $(N :_R M)$ is a 2-absorbing primary ideal of R.

Proof. (a) Let $a, b, c \in R$ and $abc \in (N :_R K)$. Then $a^t cK \subseteq N$ for some positive integer t or $b^s cK \subseteq N$ for some positive integer s or $abM \subseteq N$ since N is a $G2$-absorbing submodule of M. Therefore, $(ac)^t K \subseteq N$ or $(bc)^s K \subseteq N$ or $abK \subseteq N$ as needed.

(b) Since N is a proper submodule of M, this follows from part (a) □

Corollary 2.8. Let N be a $G2$-absorbing submodule of an R-module M. Then $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of R.

Proof. By Proposition 2.7 (b), $(N :_R M)$ is a 2-absorbing primary ideal of R. Thus, by [4, Theorem 2.2], we have $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of R. $\square$

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5].

Corollary 2.9. Let M be a multiplication R-module. If N is a $G2$-absorbing submodule of M such that $\sqrt{(N :_R M)} = (N :_R M)$, then N is a 2-absorbing submodule of M.

Proof. By Proposition 2.7 (b), $(N :_R M)$ is a 2-absorbing primary ideal of R. Thus $\sqrt{(N :_R M)} = (N :_R M)$ is a 2-absorbing ideal of R by [4, 2.2.]. Now the result follows from [2, 3.9]. $\square$

Let N be a submodule of an R-module M. The intersection of all prime submodules of M containing N is said to be the (*prime*) *radical* of N and denote by $rad(N)$. In case N does not contained in any prime submodule, the radical of N is defined to be M [10].

A proper submodule N of an R-module M is said to be a *2-absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$, and $abm \in N$, then $am \in rad(N)$ or $bm \in rad(N)$ or $ab \in (N :_R M)$ [11].

Theorem 2.10. *Let M be a multiplication R-module and N be a $G2$-absorbing submodule of M. Then N is a 2-absorbing primary submodule of M.*

Proof. Let $a, b \in R$, $m \in M$, and $abm \in N$. Then we have $a^t m \in N$ for some positive integer t or $b^s m \in N$ for some positive integer s or $abM \subseteq N$. If $abM \subseteq N$, then we are done. Suppose that $a^t m \in N$ for some positive integer t. As M is a multiplication R-module, $Rm = IM$ for some ideal I of R. Thus $a^t IM \subseteq N$. This implies that $Ia \subseteq \sqrt{(N :_R M)}$. Thus

$$aRm = aIM \subseteq \sqrt{(N :_R M)}M \subseteq (rad(N) :_R M)M \subseteq rad(N).$$

Hence $am \in rad(N)$, as needed. $\square$

Corollary 2.11. Let M be a finitely generated multiplication R-module. If N is a $G2$-absorbing submodule of M, then $rad(N)$ is a 2-absorbing submodule of M.

Proof. By Theorem 2.10, N is a is a 2-absorbing primary submodule of M. Now the result follows from [11, 2.6]. $\square$

Proposition 2.12. Let N be a $G2$-absorbing submodule of an R-module M. Then $(N :_M r)$ is a $G2$-absorbing submodule of M containing N for any $r \in R \setminus (N :_R M)$.

Proof. Let $r \in R \setminus (N :_R M)$. Suppose that $a, b \in R$ and $m \in M$ such that $abm \in (N :_M r)$. Then $rabm \in N$. Since N is a $G2$-absorbing submodule of M, either $a^t rm \in N$ or $b^s rm \in N$ for some $t, s \in \mathbb{N}$ or $ab \in (N :_R M)$. Thus $a^t m \in (N :_M r)$ or $b^s m \in (N :_M r)$ or $ab \in (N :_R M) \subseteq ((N :_M r) :_R M)$ as required. $\square$

Proposition 2.13. Let M and $\acute{M}$ be R-modules and $f : M \to \acute{M}$ be an epimorphism. Then we have the following.

(a) If N is a $G2$-absorbing submodule of M such that $ker(f) \subseteq N$, then $f(N)$ is a $G2$-absorbing submodule of $\acute{M}$.

(b) If $\acute{N}$ is a $G2$-absorbing submodule of $\acute{M}$, then $f^{-1}(\acute{N})$ is a $G2$-absorbing submodule of M.

Proof. (a) If $f(N) = \acute{M}$, then

$$Ker(f) + N = f^{-1}(f(N)) = f^{-1}(\acute{M}) = f^{-1}(f(M)) = M.$$

Thus $N = M$ a contradiction. Hence $f(N) \neq \acute{M}$. Now let $a, b \in R$ and $y \in \acute{M}$ such that $aby \in f(N)$. Then there exists $n \in N$ such that $aby = f(n)$. Since f is an epimorphism, we have $f(m) = y$ for some $m \in M$. Thus $abf(m) = f(n)$. This implies that $f(abm - n) = 0$ which gives that $abm - n \in ker(f) \subseteq N$. So $abm \in N$. Since N is a $G2$-absorbing submodule of M, $a^t m \in N$ or $b^s m \in N$ for some $t, s \in \mathbb{N}$ or $ab \in (N :_R M)$. Therefore, $a^t y \in f(N)$ or $b^s y \in f(N)$ or $ab \in (f(N) :_R \acute{M})$. Thus $f(N)$ is a $G2$-absorbing submodule of $\acute{M}$.

(b) If $f^{-1}(\acute{N}) = M$, then

$$f(M) \cap \acute{N} = f(f^{-1}(\acute{N})) = f(M) = \acute{M}.$$

Thus $\acute{N} = \acute{M}$ a contradiction. Hence $f^{-1}(\acute{N}) \neq M$. Now let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(\acute{N})$. Then $abf(m) \in f(f^{-1}(\acute{N})) = \acute{N}$. Since $\acute{N}$ is a $G2$-absorbing submodule of M, $a^t f(m) \in \acute{N}$ or $b^s f(m) \in \acute{N}$ for some $t, s \in \mathbb{N}$ or $ab\acute{M} \subseteq \acute{N}$. Therefore, $a^t m \in f^{-1}(\acute{N})$ or $b^s m \in f^{-1}(\acute{N})$ or $abM \subseteq f^{-1}(\acute{N})$. Thus $f^{-1}(\acute{N})$ is a $G2$-absorbing submodule of M. $\square$

Recall that the set of zero divisors of an R-module M; denoted by $Z(M)$ is defined by $Z(M) = \{r \in R : \exists 0 \neq x \in M, rx = 0\}$.

Theorem 2.14. *Let S be a multiplicatively closed subset of R and $S^{-1}M$ be the module of fraction of an R-module M. Then the we have the following.*

(a) If N is a $G2$-absorbing submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a $G2$-absorbing submodule of $S^{-1}M$.

(b) If $S^{-1}N$ is a $G2$-absorbing submodule of $S^{-1}M$ such that $Z(M/N) \cap S = \emptyset$, then N is a $G2$-absorbing submodule of M.

Proof. (a) Assume that $a, b \in R$, $s, t, l \in S$, $m \in M$ and $(a/s)(b/t)(m/l) \in S^{-1}N$ which implies $uabm \in N$ for some $u \in S$. Since N is a $G2$-absorbing submodule of M, $a^p um \in N$ or $b^q um \in N$ for some $p, q \in \mathbb{N}$ or $ab \in (N :_R M)$. Hence $(a/s)^p(m/l) = (a^p mu)/(s^p lu) \in S^{-1}N$ or $(b/t)^q(m/l) = (b^q mu)/(t^q lu) \in S^{-1}N$ or $ab/st \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. So, $S^{-1}N$ is a $G2$-absorbing submodule of $S^{-1}M$.

(b) First note that $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ because $Z(M/N) \cap S = \emptyset$. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$. Then $abm/1 \in S^{-1}N$. Since $S^{-1}N$ is a $G2$-absorbing submodule of $S^{-1}M$, either $(a/1)^p(m/1) \in S^{-1}N$ or $(b/1)^q(m/1) \in S^{-1}N$ for some $p, q \in \mathbb{N}$ or $ab/1 \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. If $ab/1 \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$, then $ab/1 \in S^{-1}(N :_R M)$ and we are done. Otherwise, there exists $s \in S$ such that $sa^p m \in N$ or there exists $t \in S$ such that $tb^q m \in N$. This implies $a^p m \in N$ or $b^q m \in N$, since $S \cap Z(M/N) = \emptyset$. Hence N is a $G2$-absorbing submodule of M. □

3 $G2$-Absorbing submodules over Noetherian rings

A submodule N of an R-module M is said to be *idempotent* if $N = (N :_R M)^2 M)$. Also, M is said to be *fully idempotent* if every submodule of M is idempotent [1]. Clearly, every fully idempotent R-module is a multiplication R-module.

Theorem 3.1. *Let R be a Noetherian ring and N be a submodule of a fully idempotent R-module M. If $(N :_R M)$ is a 2-absorbing primary ideal of R, then N is a $G2$-absorbing submodule of M.*

Proof. Let $a, b \in R$, K be a submodule of M, and $abK \subseteq N$. Then we have $ab(K :_R M)M \subseteq N$. Thus by [4, 2.18], either $a(K :_R M)M \subseteq \sqrt{(N :_R M)}$ or $b(K :_R M)M \subseteq \sqrt{(N :_R M)}$ or $ab \in (N :_R M)$ since $(N :_R M)$ is a 2-absorbing primary ideal of R. If $ab \in (N :_R M)$, then we are done. Otherwise, since R is Noetherian, $(a(K :_R M))^t M \subseteq N$ for some positive integer t or $(b(K :_R M))^s M \subseteq N$ for some positive integer s. Thus $(a(K :_R M))^t M \subseteq N$ or $(b(K :_R M))^s M \subseteq N$, then $a^t(K :_R M)^t M \subseteq (N :_R M)M = N$ or $b^s(K :_R M)^s M \subseteq (N :_R M)M = N$ because M is a multiplication R-module. Hence, $a^t K \subseteq N$ or $b^s K \subseteq N$ since M is a fully idempotent R-module. Therefore, N is a $G2$-absorbing submodule of M. □

The following example shows that Theorem 3.1 (a) is not satisfied in general.

Example 3.2. The $\mathbb{Z}$-module $M = \mathbb{Q}$ is not a fully idempotent $\mathbb{Z}$-module. Set $N = \mathbb{Z}$. Then we have $3.2.(1/6) \in \mathbb{Z}$ while $3^i.(1/6) \notin \mathbb{Z}$, $2^j.(1/6) \notin \mathbb{Z}$, and $2.3 \notin (\mathbb{Z} :_{\mathbb{Z}} \mathbb{Q}) = 0$ for all $i, j \in \mathbb{N}$. Thus the submodule N of M is not $G2$-absorbing submodule. But $(N :_{\mathbb{Z}} M) = 0$ is a 2-absorbing primary ideal of $\mathbb{Z}$.

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Lemma 3.3. Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M_i is a fully idempotent R_i-module, for $i = 1, 2$ if and only if M is a fully idempotent R-module.

Proof. First suppose that M is a fully idempotent R-module and N_1 is a submodule of an R_1-module M_1. Then $N = N_1 \times 0$ is a submodule of M. Thus $N = (N :_R M)^2 M = (N_1 :_{R_1} M_1)^2 M_1 \times (0 :_{R_2} M_2)^2 M_2$. Hence $N_1 = (N_1 :_{R_1} M_1)^2 M_1$. Therefore, M_1 is a fully idempotent R_1-module. Similarly, M_2 is a fully idempotent R_2-module. Conversely, let N be a submodule of M. Then $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. By assumption, $N_i = (N_i :_{R_i} M_i)^2 M_i$ for $i = 1, 2$. So

$$N = (N_1 :_{R_1} M_1)^2 M_1 \times (N_2 :_{R_2} M_2)^2 M_2 = (N :_R M)^2 M,$$

as request. $\square$

Theorem 3.4. *Let $R = R_1 \times R_2$ be a Noetherian ring and $M = M_1 \times M_2$, where M_1 is a fully idempotent R_1-module and M_2 is a fully idempotent R_2-module. Then we have the following.*

(a) A proper submodule K_1 of M_1 is a $G2$-absorbing submodule if and only if $N = K_1 \times M_2$ is a $G2$-absorbing submodule of M.

(b) A proper submodule K_2 of M_2 is a $G2$-absorbing submodule if and only if $N = M_1 \times K_2$ is a $G2$-absorbing submodule of M.

(c) If K_1 is a primary submodule of M_1 and K_2 is a primary submodule of M_2, then $N = K_1 \times K_2$ is a $G2$-absorbing submodule of M.

Proof. (a) Let K_1 be a $G2$-absorbing submodule of M_1. Then $(K_1 :_{R_1} M_1)$ is a 2-absorbing primary ideal of R_1 by Proposition 2.7. Now since $(N :_R$

$M) = (K_1 :_{R_1} M_1) \times R_2$, we have $(N :_R M)$ is a 2-absorbing primary ideal of R by [4, 2.23]. Thus the result follows from Lemma 3.3 and Theorem 3.1. Conversely, let $N = K_1 \times M_2$ be a $G2$-absorbing submodule of M. Then $(N :_R M) = (K_1 :_{R_1} M_1) \times R_2$ is a primary ideal of R by Proposition 2.7. Thus $(K_1 :_{R_1} M_1)$ is a primary ideal of R_1 by [4, 2.23]. Hence by Theorem 3.1, K_1 is a $G2$-absorbing submodule of M_1.

(b) This is proved similar to the part (a).

(c) Let K_1 be a primary submodule of M_1 and K_2 be a primary submodule of M_2. Then $(K_1 :_{R_1} M_1)$ and $(K_2 :_{R_2} M_2)$ are primary ideals of R_1 and R_2, respectively. Now since $(N :_R M) = (K_1 :_{R_1} M_1) \times (K_2 :_{R_2} M_2)$, we have $(N :_R M)$ is a 2-absorbing primary ideal of R by [4, 2.23]. Thus the result follows from Theorem 3.1. □

Theorem 3.5. *Let $R = R_1 \times R_2$ be a Noetherian ring and $M = M_1 \times M_2$ be a fully idempotent R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a proper submodule of M. Then the following conditions are equivalent:*

(a) N is a $G2$-absorbing submodule of M;

(b) Either $N_1 = M_1$ and N_2 is a $G2$-absorbing submodule of M_2 or $N_2 = M_2$ and N_1 is a $G2$-absorbing submodule of M_1 or N_1, N_2 are primary submodules of M_1, M_2, respectively.

Proof. $(a) \Rightarrow (b)$. Let $N = N_1 \times N_2$ be a $G2$-absorbing submodule of M. Then $(N :_R M) = (K_1 :_{R_1} M_1) \times (K_2 :_{R_2} M_2)$ is a 2-absorbing primary ideal of R by Proposition 2.7. By [4, 2.23], we have $(K_1 :_{R_1} M_1) = R_1$ and $(K_2 :_{R_2} M_2)$ is a 2-absorbing primary ideal of R_2 or $(K_2 :_{R_2} M_2) = R_2$ and $(K_1 :_{R_1} M_1)$ is a 2-absorbing primary ideal of R_1 or $(K_1 :_{R_1} M_1)$ and $(K_2 :_{R_2} M_2)$ are primary ideals of R_1 and R_2, respectively. Suppose that $(K_1 :_{R_1} M_1) = R_1$ and $(K_2 :_{R_2} M_2)$ is a 2-absorbing primary ideal of R_2. Then $N_1 = M_1$ and N_2 is a $G2$-absorbing submodule of M_2 by Theorem 3.4 and Lemma 3.3. Similarly if $(K_2 :_{R_2} M_2) = R_2$ and $(K_1 :_{R_1} M_1)$ is a 2-absorbing primary ideal of R_1. Then $N_2 = M_2$ and N_1 is a $G2$-absorbing submodule of M_1. If the last case hold, then as M_1 (resp. M_2) is a multiplication R_1-(resp. R_2-) module, N_1 (resp. N_2) is a primary submodule of M_1 (resp. M_2) by [8, Corollary 2].

$(b) \Rightarrow (a)$. This can be proved easily by using Theorem 3.4. □

Theorem 3.6. *Let R be a Noetherian ring, N be a $G2$-absorbing submodule of an R-module M, and $m \in M \setminus N$. Then $\sqrt{(N :_R m)}$ is a prime ideal of R or there exists an element $a \in R$ such that $\sqrt{(N :_R a^n m)}$ is a prime ideal of R for some positive integer n.*

Proof. By Corollary 2.8, $\sqrt{(N :_R M)}$ is a 2-absorbing ideal of R, therefore by [4, Theorem 2.3], we have either $\sqrt{(N :_R M)} = \mathrm{p}$ or $\sqrt{(N :_R M)} = p \cap q$, where p and q are distinct prime ideals of R. Suppose that $\sqrt{(N :_R M)} = p$. Then $p = \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$. We show that $\sqrt{(N :_R m)}$ is a prime ideal of R. Let $ab \in \sqrt{(N :_R m)}$ for some $a, b \in R$. Then $(ab)^n \in (N :_R m)$ implies $(ab)^n m \in N$. As N is a $G2$-absorbing submodule of M, then either $a^{nt}m \in N$ or $b^{ns}m \in N$ for some $t, s \in \mathbb{N}$ or $(ab)^n \in (N :_R M)$. If $a^{nt}m \in N$ or $b^{ns}m \in N$, then $a \in \sqrt{(N :_R m)}$ or $b \in \sqrt{(N :_R m)}$. If $(ab)^n \in (N :_R M)$, then $ab \in p$. Since p is prime ideal of R, then either $a \in p \subseteq \sqrt{(N :_R m)}$ or $b \in p \subseteq \sqrt{(N :_R m)}$. Therefore, $\sqrt{(N :_R m)}$ is a prime ideal of R. Now suppose that $\sqrt{(N :_R M)} = p \cap q$. If $p \subseteq \sqrt{(N :_R m)}$, then by previous argument, we have $\sqrt{(N :_R m)}$ is a prime ideal of R. If $p \nsubseteq \sqrt{(N :_R m)}$, then there exists $a \in p \setminus \sqrt{(N :_R m)}$. Also,

$$pq \subseteq \sqrt{pq} = \sqrt{p \cap q} = \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}.$$

Now since R is Noetherian, there exists $n \in \mathbb{N}$ such that $(pq)^n \subseteq (N :_R m)$. This implies that $q \subseteq \sqrt{(N :_R a^n m)}$ and the result follows by a similar argument. □

Now, we study $G2$-absorbing avoidance Theorem for submodules. We first define an efficient covering of submodules: let $N, N_1, N_2, ..., N_n$ be submodules of an R-module M. An efficient covering of N is a covering $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ in which no N_k (where $1 \leq k \leq n$) satisfies $N \subseteq N_k$. In other words, a covering $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ is efficient if no N_k is superfluous. Analogously, we say that $N = N_1 \cup N_2 \cup ... \cup N_n$ is an efficient union if none of the N_i may be excluded. Any cover or union consisting of submodules of M can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

To proceed further, we require the following lemma.

Lemma 3.7. [9, Lemma 2.1]. Let $N = N_1 \cup ... \cup N_n$ be an efficient union of submodules of an R-module M for $n > 1$. Then $\cap_{j \neq k} N_j = \cap_{j=1}^{n} N_j$ for all k.

Theorem 3.8. *Let R be a Noetherian ring and $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ be an efficient covering consisting of submodules of an R-module M, where $n > 2$. If $\sqrt{(N_j :_R M)} \nsubseteq \sqrt{(N_k :_R m)}$ for all $m \in M \setminus N_k$ whenever $j \neq k$, then no N_i is a $G2$-absorbing submodule of M.*

Proof. Suppose N_k is a $G2$-absorbing submodule of M for some $1 \leq k \leq n$, and look for a contradiction. Since $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ is an efficient covering, $N \nsubseteq N_k$, so there exists an element $m_k \in N \setminus N_k$. It is clear that $N = (N_1 \cap N) \cup (N_2 \cap N) \cup ... \cup (N_n \cap N)$ is an efficient union. By Lemma 3.7, we have $\cap_{j \neq k}(N \cap N_j) \subseteq N \cap N_k$. By using Theorem 3.6, we have either $\sqrt{(N_k :_R m_k)}$ is a prime ideal of R or there exists $a \in R$ such that $\sqrt{(N_k :_R a^n m_k)}$ is a prime ideal of R. First, suppose that $\sqrt{(N_k :_R m_k)}$ is a prime ideal. By the given hypothesis $\sqrt{(N_j :_R M)} \nsubseteq \sqrt{(N_k :_R m_k)}$ for $j \neq k$. So, there exists $s_j \in \sqrt{(N_j :_R M)}$ but $s_j \notin (\sqrt{N_k :_R m_k})$, where $j \neq k$. This implies that $s_j^{n_j} \in (N_j :_R M)$ but $s_j^{n_j} \notin (N_k :_R m_k)$ where $j \neq k$ and $n_j \in \mathbb{N}$. Let $s = \prod_{j \neq k} s_j$. Then $s \in \sqrt{(N_j :_R M)}$ but $s \notin \sqrt{(N_k :_R m_k)}$ where $j \neq k$. Therefore, $s^m \in (N_j :_R M)$ for all $j \neq k$ but $s^m \notin (N_k :_R m_k)$, where $m = max\{n_1, n_2, ..., n_{k-1}, n_{k+1}, ..., n_n\}$. Thus $s^m m_k \in \cap_{j \neq k}(N \cap N_j) \setminus (N \cap N_k)$, since $s^m m_k \in N \cap N_k$ implies $s^m \in (N_k :_R m_k)$, a contradiction. So, no N_k is a $G2$-absorbing submodule of M. Now, consider the case when $\sqrt{(N_k :_R a^n m_k)}$ is a prime ideal, where n is positive integer and $a \in R$. Clearly, $s_j \in \sqrt{(N_j :_R M)}$ but $s_j \notin \sqrt{(N_k :_R a^n m_k)}$, where $j \neq k$. Therefore, $s^m a^n m_k \in \cap_{j \neq k}(N \cap N_j) \setminus (N \cap N_k)$, since $s^m a^n m_k \in N \cap N_k$ implies $s^m \in (N_k :_R a^n m_k)$, a contradiction. So, no N_k is $G2$-absorbing submodule of M. □

Theorem 3.9. *($G2$-Absorbing Avoidance Theorem). Let R be a Noetherian ring and $N, N_1, ..., N_n$ ($n \geq 2$) be submodules of an R-module M such that at most two of $N_1, N_2, ..., N_n$ are not $G2$-absorbing submodules. If $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ and $\sqrt{(N_j :_R M)} \nsubseteq \sqrt{(N_k :_R m)}$ for all $m \in M \setminus N_k$ whenever $j \neq k$, then $N \subseteq N_i$ for some $1 \leq i \leq n$.*

Proof. If $n = 2$, then it is obvious. Now, take $n > 2$ and $N \nsubseteq N_i$ for all $1 \leq i \leq n$. Then $N \subseteq N_1 \cup N_2 \cup ... \cup N_n$ is an efficient covering. Using Theorem 3.8, no N_i is a $G2$-absorbing submodule, which is a contradiction. Hence $N \subseteq N_i$ for some $1 \leq i \leq n$. □

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